

Unit 5: Multivariable Calculus Extension Question

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Theorem: Let $\vec{F} : \mathbb{R}^n \rightarrow \mathbb{R}^n$, and $\vec{c} : [a, b] \subseteq \mathbb{R} \rightarrow \mathbb{R}^n$. If $\mathbf{D}_{\vec{F}}$ is diagonal and exists for every vector in $\text{dom}(\vec{F})$, then $\int_a^b \vec{F} \cdot d\vec{s}$ is path-independent (meaning that as long as one parameterization of $\vec{c}$ is such that given two constant vectors $\vec{a}$ and $\vec{b}$, $\vec{c}(a) = \vec{a}$ and $\vec{c}(b) = \vec{b}$, any other parameterization that matches those two end-point values for $t = a$ and $t = b$ will have the same vector-line integral with $\vec{F}$).

Proof. If $\mathbf{D}_{\vec{F}}$ is diagonal, then each component function of $\vec{F}$ can be written as a single-variable function from $\mathbb{R} \rightarrow \mathbb{R}$. Thus we can represent $\vec{F}$ as:

$$\vec{F}(x_1, x_2, x_3, \dots, x_n) = (f_1(x_1), f_2(x_2), f_3(x_3), \dots, f_n(x_n))$$

where $f_i : \mathbb{R} \rightarrow \mathbb{R}$.

Similarly, we can write $\vec{c}$ as:

$$\vec{c}(t) = (x_1(t), x_2(t), x_3(t), \dots, x_n(t))$$

where $x_i : \mathbb{R} \rightarrow \mathbb{R}$.

The path integral of $\vec{F}$ along $\vec{c}$ is given by

$$\int_{\vec{c}} \vec{F} \cdot d\vec{s} = \int_a^b \vec{F}(\vec{c}(t)) \cdot \vec{c}'(t) dt$$

Expanding the dot product we get:

$$\int_a^b f_1(x_1(t)) \cdot \vec{x}_1'(t) + f_2(x_2(t)) \cdot \vec{x}_2'(t) + \dots + f_n(x_n(t)) \cdot \vec{x}_n'(t) dt$$

Notice that each term of the integrand is an application of the chain rule when taking the derivative with respect to t of $F_i(x_i(t))$, where F_i

represents an anti-derivative of f_i with respect to dx_i . Using this observation, the integral can now be expressed as:

$$\begin{aligned} & \left. (F_1(x_1(t)) + F_2(x_2(t)) + \dots + F_n(x_n(t))) \right|_a^b \\ &= (F_1(x_1(b)) + F_2(x_2(b)) + \dots + F_n(x_n(b))) - (F_1(x_1(a)) + F_2(x_2(a)) + \dots + F_n(x_n(a))) \end{aligned}$$

Remember that $\vec{F}$ will always be the same function regardless of the parametrization of $\vec{c}$, therefore making the respective anti-derivatives of the components of $\vec{F}$ with respect to x_i the same as well (regardless of the parametrization of $\vec{c}$). Thus, the only values affecting the result of the vector-line integral are $x_1(a), x_2(a), \dots, x_n(a)$ and $x_1(b), x_2(b), \dots, x_n(b)$, or in other words, $\vec{c}(a)$ and $\vec{c}(b)$ are the only values which affect the result of the vector-line integral.

Thus, given two paths $\vec{c}_1 : [a, b] \subseteq \mathbb{R} \rightarrow \mathbb{R}^n, \vec{c}_2 : [a, b] \subseteq \mathbb{R} \rightarrow \mathbb{R}^n$, if $\vec{c}_1(a) = \vec{c}_2(a)$ and $\vec{c}_1(b) = \vec{c}_2(b)$,

$$\int_{\vec{c}_1} \vec{F} \cdot d\vec{s} = \int_{\vec{c}_2} \vec{F} \cdot d\vec{s}$$

Even if $\forall t \in S \subseteq (a, b), \vec{c}_1(t) \neq \vec{c}_2(t)$ (notice that S represents a set which is either a subset of or equal to (a, b)).

Therefore, by the definition of path-independence, if $\mathbf{D}_{\vec{F}}$ is diagonal and exists for every vector in $\text{dom}(\vec{F})$, then $\int_a^b \vec{F} \cdot d\vec{s}$ is path-independent. $\square$

Supplementary Desmos graph for a demonstration of this theorem:

<https://www.desmos.com/calculator/blvdubmjjz>