

Problem 59

Let $\vec{c}(t)$ be a path in $\mathbb{R}^3$ on the domain $t \in [a, b]$.

(a) Define a function $\alpha : [a, b] \rightarrow [c, d]$ be a function that has the following properties:

- $\alpha(a) = c$ and $\alpha(b) = d$
- α is differentiable on (a, b) and continuous on $[a, b]$.
- $\alpha'(t) > 0$ for all $t \in (a, b)$.

Explain why α must be bijective and hence invertible. I will refer to such a function α as a faithful parameter.

Answer: Firstly, note that α is continuous over the closed interval $[a, b]$ and that $\alpha(a) = c$ and $\alpha(b) = d$, thus by the Intermediate Value Theorem we can conclude that there must exist at least one mapping to every element in $[c, d]$ between a and b for α , thus by the definition of surjectivity, α is surjective. Secondly, note that α is always increasing over the open interval (a, b) . Since α is always increasing, there cannot exist any repeated mappings to any one value in $[c, d]$ for any values between a and b inputted into α (since we will always be at least "slightly" increasing our output for every infinitesimally small increment in $[a, b]$, and c is our minima while d is our maxima over this interval). Thus, by the definition of injectivity (exactly one value in the domain will correspond to a value in the codomain), we can conclude that α must also be injective. Therefore, since α is both surjective and injective, it must be by definition *bijective* (and thus invertible) as well.

(b) We define a new path $\vec{d} : [c, d] \rightarrow \mathbb{R}^3$ by $\vec{d}(u) = \vec{c}(\alpha^{-1}(u))$. Show that the image curves of $\vec{c}$ and $\vec{d}$ are the same.

Answer: This can easily be done by noting that $\vec{d}$ is simply a transformation of the parameter of $\vec{c}$. By the definition of this type of transformation, the actual image curve itself must be equal (warranting that the same domain is provided by that transformation itself). Since α^{-1} (using our results from the previous section) must have a codomain of $[a, b]$ given that the full domain interval of $[c, d]$ is applied through the

equivalent domain of $\vec{d}$, we can safely conclude that the resulting image curves of $\vec{d}$ and $\vec{c}$ must be equivalent, as they will be the same "base" curve ($\vec{c}$) over the same domain ($[a, b]$).

- (c) Show that the paths $\vec{c}$ and $\vec{d}$ have the same arc length.

Answer: Note that arc length is defined as the distance along the curve on some interval $[x, y]$ for t . Note that for $\vec{c}$, the arc length is simply $\int_a^b \|\vec{c}'(t)\| dt$. Now, let us think about the arc length of $\vec{d}$. For $\vec{d}$, we'll be analyzing under the domain $[c, d]$, thus the arc length of $\vec{d}$ will be $\int_c^d \|\vec{d}'(u)\| du$. Substituting $\vec{d}(u) = \vec{c}(\alpha^{-1}(u))$ and applying the chain rule gives us:

$$\begin{aligned} \int_c^d \|\vec{c}'(\alpha^{-1}(u))\alpha^{-1}'(u)\| du &= \int_c^d \|\vec{c}'(\alpha^{-1}(u))\| \alpha^{-1}'(u) du \\ &= \int_c^d \|\vec{c}'(t)\| dt \end{aligned} \quad \text{Where } t = \alpha^{-1}(u)$$

Note that we can resolve the "u-substitution" by changing the interval of integration. Since $\alpha^{-1}(d) = b$ and $\alpha^{-1}(c) = a$, we can now write our final integral:

$$\int_a^b \|\vec{c}'(t)\| dt$$

The above integral is equivalent to the arc length of $\vec{c}$, thus $\vec{c}$ and $\vec{d}$ have equivalent arc lengths.

- (d) Show that the function $s(t) = \int_a^t \|\vec{c}'(u)\| du$ is a faithful parameter.

Firstly, let us define the domain and codomain of s :

$$s : [a, b] \rightarrow [0, \text{arclength of } \vec{c}]$$

Nextly, note that $s(a) = \int_a^a \|\vec{c}'(u)\| du = 0$ since the interval of integration is of no length. Furthermore, $s(b) = \int_a^b \|\vec{c}'(u)\| du = \text{arclength of } \vec{c}$ (by definition of arclength). Thus, the first two conditions to meet a faithful parameter have been met. For our other conditions of differentiability and subsequent continuity, s must be differentiable and therefore continuous over $[a, b]$ because the derivative of $s(t)$ is guaranteed to be

defined due to the Fundamental Theorem of Calculus (derivative of an integral varying its interval of integration is just the integrand as its own function).

Finally, $s'(t) > 0$ for all $t \in [a, b]$, as the definition of $s'(t) = \|\vec{c}'(t)\|$ for all $[a, b]$, and a magnitude is by definition always positive.

Thus, since s fits all the the criteria above, s must be a faithful parameter.

(e) Let $\vec{d}(s)$ be the curve $\vec{c}(s^{-1}(t))$. Show that $\|\vec{d}'(s)\| = 1$ for all $s \in [c, d]$.

$$\begin{aligned}
 \|\vec{d}'(s)\| &= \|\vec{c}'(s^{-1}(t))s^{-1}'(t)\| \\
 &= \left\| \vec{c}'(s^{-1}(t)) \cdot \frac{1}{s'(s^{-1}(t))} \right\| \\
 &= \left\| \vec{c}'(s^{-1}(t)) \cdot \frac{1}{\vec{c}'(s^{-1}(t))} \right\| \quad \text{Note that } s'(t) = \|\vec{c}'(t)\| \\
 &= \|1\| \\
 &= 1
 \end{aligned}$$

Thus, $\|\vec{d}'(s)\| = 1$ for all $s \in [c, d]$.