

Unit 4: Multivariable Calculus Extension Question #48

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Question: Let $\Phi : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be the inverse of the map $T(x, y) = (xy, x^2y)$. Let D be the following domain:

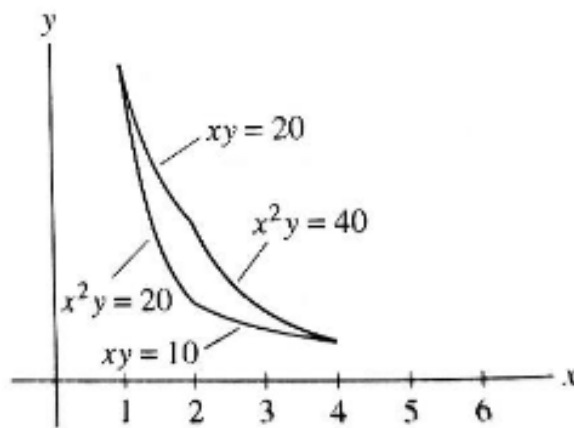


FIGURE 19

Show by applying a change of variables that

$$\iint_D e^{xy} dA = \int_{10}^{20} \left(\int_{20}^{40} e^u v^{-1} dv \right) du$$

and evaluate the result.

Deriving the iterated integral: Firstly, let us determine the area differential, dA^* , or as it is defined by the Change of Variables Theorem, $|\det(D_\Phi)|$. Note that T is by definition invertible. Thus, if the area differential created by the transformation T may be represented by $\det(D_T)$, the inverse function of T , which will transform these “warped” areas back into their original state (as $\Phi \circ T = I$, where I is the identity transformation in $\mathbb{R}^2$ defined by $I(x, y) = (x, y)$) must therefore have an area differential equivalent to the *multiplicative inverse* of $\det(D_T)$.

More rigorously, $\Phi \circ T = I$ implies that $D_\Phi D_T = D_I$ by the Chain Rule. Therefore, we can write that

$$\det(D_\Phi D_T) = \det(D_I).$$

Using determinant rules, we can say that

$$\det(D_\Phi) \det(D_T) = 1$$

so

$$\det(D_\Phi) = \det(D_T)^{-1}.$$

We may now calculate $\det(D_T)^{-1}$ to obtain $\det(D_\Phi)$ (which we will then take the absolute value of as is necessitated by the Change of Variables theorem...signed area is not very useful to us here as we don't care about how the orientation changed):

$$D_T = \begin{bmatrix} y & x \\ 2xy & x^2 \end{bmatrix}$$

$$\det(D_T) = x^2y - 2x^2y = -x^2y$$

$$\det(D_\Phi) = \det(D_T)^{-1} = (-x^2y)^{-1} = \frac{1}{-x^2y}$$

Now the area differential can be obtained:

$$dA^* = \left| \frac{1}{-x^2y} \right| = \left| \frac{1}{x^2y} \right|$$

Now, we must obtain the pre-image, D^* , of D in order to find our integration bounds. Note that D^* is defined as $D^* \subseteq \mathbb{R}^2$ such that $\Phi(D^*) = D$. I would first like to explain geometrically and intuitively how we may go about finding D^* , and then we can define D^* using more rigorous means. In trying to find D^* , we are trying to find a rectangle in the uv -plane such that by applying our transformation Φ it will form the diagram shown above. The simplest method is to first consider the 4 corners of the region D . Firstly, let us examine the bottom left corner of the diagram. In the xy -plane, the corner can be represented as either $(2, 5)$ or as the solution to the system defined by $(xy, x^2y) = (10, 20)$, or $T(x, y) = (10, 20)$. If we apply Φ to both sides of this equation, we get $\Phi(T(x, y)) = \Phi(10, 20) \rightarrow (x, y) = \Phi(10, 20)$. Thus, this corner can be represented via $\Phi(10, 20)$. The same process may then be applied to the other three corners to obtain $\Phi(10, 40)$, $\Phi(20, 40)$, and $\Phi(20, 20)$. Note that the edges of D are themselves just defined by fixing one of the component functions of T to one of its boundaries and allowing the other

component to be any value between its own assigned boundary (either in $[10, 20]$ for T_x or $[20, 40]$ for T_y). This leads one to intuitively see that $D^* = [10, 20] \times [20, 40]$.

In more rigorous terms, firstly let A be the set defined by $A = \{T(x, y) \mid (x, y) \in D\}$. $\exists(u, v) \in A$ and $\exists(x, y) \in D$ by the fact that D is defined to be non-empty. By the definition of T and the diagram of D , we can see that:

$$A = \{T(x, y) \mid (x, y) \in D\} = [10, 20] \times [20, 40],$$

thus, if $(u, v) \in A$,

$$(u, v) \in [10, 20] \times [20, 40].$$

Now, let us perform the transformation Φ unto $A = [10, 20] \times [20, 40]$, which means that every member of A will have Φ applied unto it, including the member (u, v) denoted above:

$$\Phi(u, v) \in \Phi([10, 20] \times [20, 40]).$$

By our original definition of A , $\forall(u, v) \in A$, $\exists(x, y) \in D$ such that $(u, v) = T(x, y)$. Thus we can write

$$\Phi(T(x, y)) \in \Phi([10, 20] \times [20, 40]).$$

However, since $\Phi \circ T = I$, we can simply write (x, y) instead:

$$(x, y) \in \Phi([10, 20] \times [20, 40])$$

for all $(x, y) \in D$. By the definition of a subset, the previous statement implies that $D \subseteq \Phi([10, 20] \times [20, 40])$ so $D \subseteq \Phi(A)$. Now let us observe the following:

$$(x, y) \in D$$

We can use the same technique from before and carefully apply the transformation $I = \Phi \circ T$ unto D , therefore applying it to all of its members. Thus,

$$\Phi(T(x, y)) \in \Phi(T(D)).$$

Using the definition of the set A , we may now substitute $T(x, y)$ with $(u, v) \in A$, thus we now have:

$$\Phi(u, v) \in \Phi(T(D))$$

for all $(u, v) \in A$. Note that since $\Phi \circ T = I$ by the definition of Φ and T we can rewrite this statement as

$$\Phi(u, v) \in D$$

for all $(u, v) \in A$. Thus, we can conclude from the definition of a subset that $\Phi(A) \subseteq D$.

Now, since it has been established that $D \subseteq \Phi(A)$ and $\Phi(A) \subseteq D$, we can conclude that $D = \Phi(A)$ by the Double-Inclusion Theorem from Linear Algebra last semester. Using the definition of D^* from earlier, we may then conclude that $D^* = A$, or, using the definition of A established earlier, that $D^* = [10, 20] \times [20, 40]$. Thus, we have achieved the same definition for D^* by both an intuitive and rigorous method.

Note that since $T(x, y) = (u, v)$ for all corresponding $(x, y) \in D$ and $(u, v) \in D^*$ (as a result of what we just proved above), we can write that $T(x, y) = (xy, x^2y) = (u, v)$ for corresponding pairs of (x, y) and (u, v) . This allows us to state that $u = xy$ and $v = x^2y$ for the purposes of substitution in our integral.

We now finally have all the tools to apply the Change of Variables Theorem unto our original volume integral:

$$\begin{aligned} \iint_D e^{xy} dA &= \iint_{D^*} e^u dA^* \\ &= \int_{10}^{20} \int_{20}^{40} e^u \left| \frac{1}{x^2y} \right| dv du \\ &= \int_{10}^{20} \int_{20}^{40} e^u |v^{-1}| dv du \end{aligned}$$

Note that since the bounds for v are both positive, it is unnecessary to include the absolute value bars for an expression that (over our region D) will always be positive anyways. Thus, finally, we can conclude that using the Change of Variables Theorem we get

$$\iint_D e^{xy} dA = \int_{10}^{20} \left(\int_{20}^{40} e^u v^{-1} dv \right) du$$

□

Evaluating the integral:

$$\begin{aligned}\int_{10}^{20} \left(\int_{20}^{40} e^u v^{-1} dv \right) du &= \int_{10}^{20} (e^u \ln |v|) \Big|_{20}^{40} du \\&= \int_{10}^{20} (e^u \ln 40 - e^u \ln 20) du \\&= \int_{10}^{20} (e^u \ln 2) du \\&= (e^u \ln 2) \Big|_{10}^{20} \\&= e^{20} \ln 2 - e^{10} \ln 2 \\&= \ln(2) (e^{20} - e^{10})\end{aligned}$$